Designing Cognitively Demanding Problems

NCCTM Conference, Greensboro, North Carolina, November 1, 2013

Activity 1: Sorting Problems by Cognitive Demand

- For each group of problems (A, B, C, and D), sort them into the lower or higher cognitive demand columns. Solve the problems as needed to determine cognitive demand.
- Write problem numbers into the corresponding cells in the table. Explain which problem features increase their cognitive demand within each group. Discuss with your group/partner whether you have discrepancies in sorting and in determining the features.
- Write your group's features for problems with a higher cognitive demand level on the poster and put it on the wall.

Table 1: Sorting Problems by Cognitive Demand

	Cognitive Demand		Which problem features make it a higher cognitive demand?
	Higher	Lower	
Group A			
Group B			
Group C			
Group D			

Problems

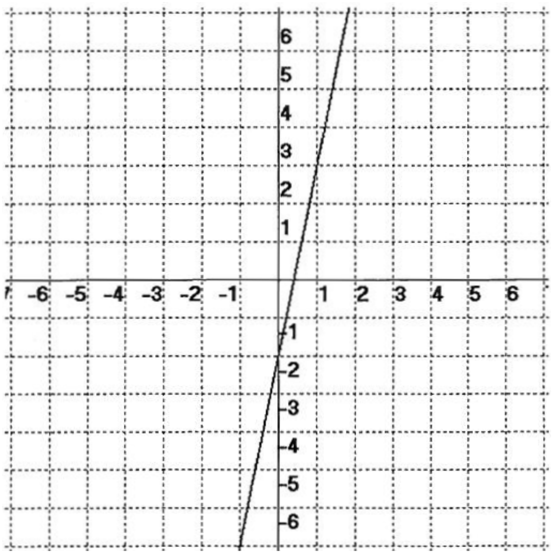
Group A

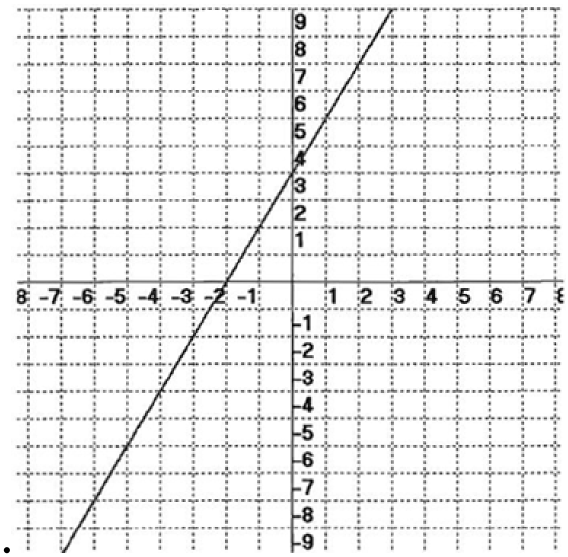
1. Solve the following system of equations by graphing. Use your graphing calculator.

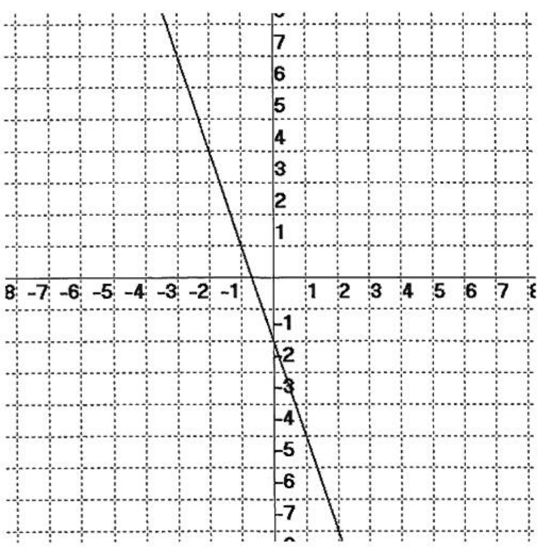
$$-2x + 4y = 34$$

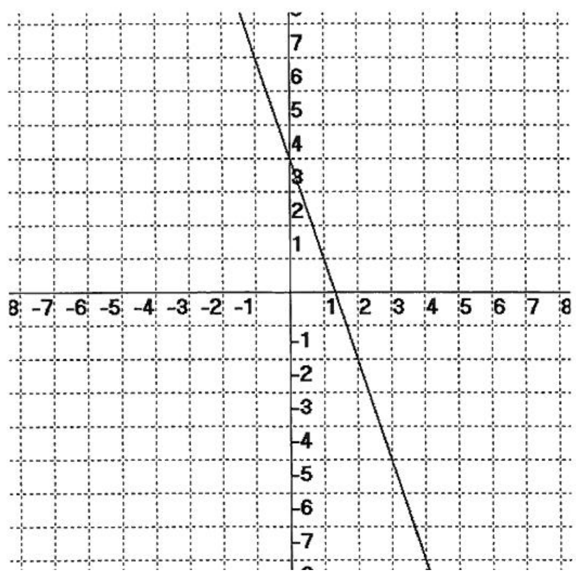
$$7x - y = -28$$

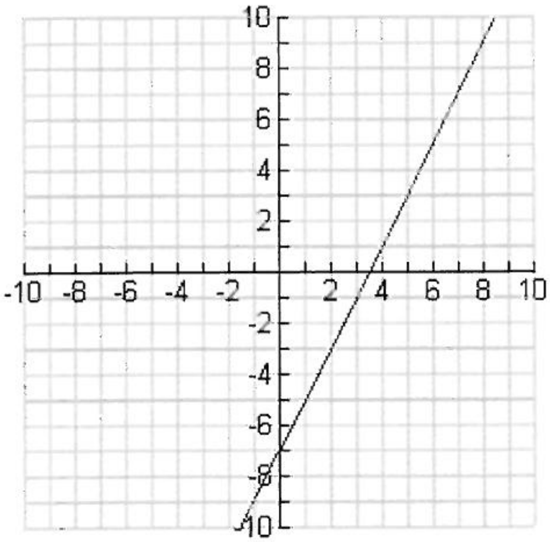
2. Consider the following linear functions represented graphically.

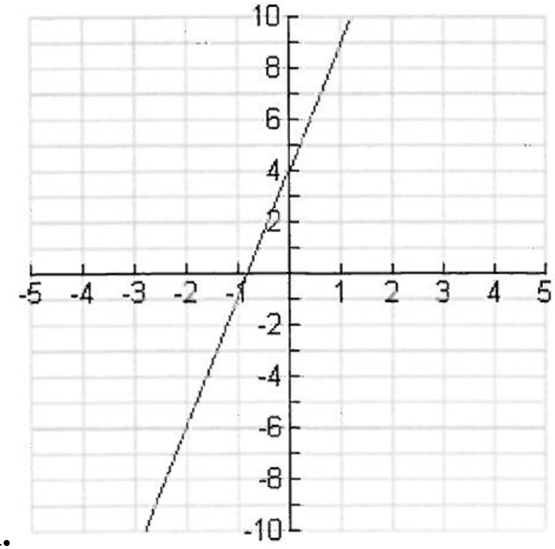
2i. 

2j. 

2k. 

2l. 

2m. 

2n. 

2a. Sort these functions into two or three groups based on common features among them.

2b. Explain the criteria for your sorting. What do the functions in each group have in common? What are some differences between functions within the same group?

Group B

3.

3a. On graph paper, draw a square with side length a, where $3 \le a \le 75$. Draw a rectangle with one side $a + 1$ and another side $a - 1$. Compare the areas of the square and the rectangle that you drew.

3b. Draw another example of the square and the rectangle fitting this description with a different side length a. Compare their areas.

3c. Draw a third example of the square and the rectangle fitting this description with a yet another side length a. Compare their areas.

3d. Observe patterns and formulate a conjecture about your observation. Write this conjecture as an algebraic equation.

3e. Will this relationship be true for any square and rectangle that fit this description? Justify your response using diagrams and algebraically.

4.

4a. Expand the expression $a^2 – b^2$ =

4b. Use this formula to mentally evaluate the following expressions:

$64^2 - 36^2$ $128^2 - 28^2$ $29^2 - 9^2$ $73^2 - 27^2$

Group C

5. Use the cross-product method to solve for x in the proportion $\frac{2x}{7} = \frac{5a}{6}$.

6.

6a. Write at least three proportions that have the cross-product $6 \cdot 2x = 5a \cdot 7$.

6b. How do these proportions relate to each other?

7. Melanie uses a cell phone plan. The plan requires her to pay a $20 monthly fee for the first 200 minutes and an additional 25¢ per minute on additional minutes. For the month of January Melanie spent $40. How many minutes did she talk?

8. Melanie and her friends are using three different cell phone plans, each of which has a flat monthly fee and a per-minute charge for additional minutes. One month, all three friends talked for 400 minutes and spent the same amount of money: $40. What could their plans be? Specify the flat fee, f and the charge per minute, c.

Group D

9. Tony is buying a used car. He will choose between two cars. The table below shows information about each car.

Car	Cost	Miles per Gallon	Estimated Immediate Repairs
Car A	$3,200	18	$700
Car B	$4,700	24	$300

Tony wants to compare the total costs of buying and using these cars.

- Tony estimates he will drive at least 200 miles per month.
- The average cost of gasoline per gallon in his area is $3.70.
- Tony plans on owning the car for 4 years.

Calculate and explain which car will cost Tony the least to buy and use.

10. Tony estimates he uses about 10 gallons of gasoline per month. If the average cost of gasoline per gallon in his area is $3.70, how much does Tony spend on gasoline per month?

Activity 2: Designing Cognitively Demanding Problems

Table 2: Some Ways to Modify Problems to Increase Cognitive Demand

1.	Design a mathematical investigation to discover an important formula or relationship.
2.	Combine problems that develop and build one idea and have students analyze, compare and contrast these problems together.
3.	Invert a problem: The answer becomes "what is given" in a problem, and "the given" becomes what has to be found/solved for. This technique creates an open-ended problem with multiple answers (and possibly solutions).
4.	Increase the number of "steps" required for solution without specifying the sequence of these steps. Require explanations/justification for the solution plan.

- Apply the techniques specified above to modify at least two problems below or another problem of your choice in order to create problems with a higher level of cognitive demand.
- Write your modified problems on a poster and put it on the wall.

A. Combine simpler problems to create a multi-step problem, ask students to check the answer to the problem using mathematical procedures:

1. A student room in the dormitory is 10 feet wide, 13 feet long, and 9 feet high. The room has a door sized 2.5x8 feet and a window sized 5x6 feet. Calculate the area of the room's walls excluding the door and the window.

2. A painter uses 1 can of paint to cover 90 ft^2 of the wall. How many cans of paint would he need to paint 350 ft^2 of the wall?

3. The student dormitory building has twelve stories. The number of student rooms on the first floor is 12, and each following floor has two more student rooms than the previous one. The rest of the space is the common space. How many student rooms total are in the building?

4. A building has 118 rooms. To paint each room's walls, the painter spends 5 hours of work time and 2 cans of paint. The painter is paid $12 per hour and a can of pain costs $8. What would be the total cost of painting all 118 rooms?

B. Revise one of the following problems so that students compare and contrast:

5. On your graphing calculator, graph the following functions:

$$y = 4x - 3 \qquad y = 4(x-3)$$

$$y = 4^x - 3 \qquad y = 4^{x-3}$$

6. a. Find the distance between points $A(2, -4)$ and $B(-1, 5)$.

b. Find the length of the diagonal of a rectangle with sides of lengths 3 cm and 9 cm.

C. Revise one of the following problems so that students solve the converse of the problem or create multiple examples:

7. Simplify $(7k+11)-(p-3k)+(13+9k)$

8. Write an equation for the following situation: Jacob bought four packs of batteries and paid $18. He got $2 in change.

9. Find the roots of the equation: $5x^2 - 5 = 0$

10. Find a distance between points $A(1, 1)$ and $B(0, -2)$.

D. Revise the following problem so that students make a generalization (Design a mathematical investigation):

11. A rectangle *ABCD* has sides 3cm and 9cm. A rectangle *KLMN* has one side three times as long and another side one third of another side of rectangle *ABCD.* How does the area of rectangle *KLMN* compare with the area of rectangle *ABCD?*

Activity 3: Looking at Mathematical Processes during Problem Solving

1. Make sense of problems and persevere in solving them.

“Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.”

- Discuss with your group: What are the two or three problems that use the greatest number of the processes listed in the table below?

- Are there any processes in the table of Mathematical Processes (that you have not yet listed) that are or could be at work in problems that you classified into the higher cognitive demand category?

The purpose of the table below is to describe discreet mathematical processes and types of mathematical work which students can engage in within a CCSSM-compliant mathematics classroom. These descriptions can be helpful in designing (or evaluating) problems and tasks that engage students in cognitively demanding types of work and in asking questions to elicit higher order types of thinking.

Table 3: Mathematical Processes and Types of Mathematical Work

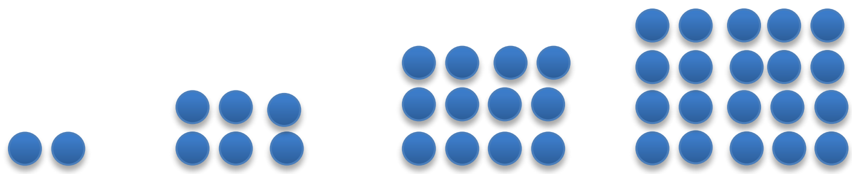
Processes Used in Problems with Lower Cognitive Demand (levels 1 [Memorization] & 2 [Procedures Without Connections] of Stein et al., 2000 classification)
Either memorizing or reproducing from memory facts, rules, formulae, or definitions (level 1)
Using a specific and obvious procedure or algorithm (obvious because it has just been shown, or because of the placement in specific chapter/unit, or because of the problem type that is determined by appearance of the problem). There is little ambiguity about what needs to be done or how to do it (level 2).
A task is focused on producing a correct answer, not on developing understanding, or developing a new solution, or any connections within or outside of mathematics. It requires no explanations or explanations that describe the steps of the procedure in sequence (level 2).
Processes Used in Problems and Tasks with Higher Cognitive Demand (levels 3 [Procedures with Connections] & 4 [Doing Mathematics] of Stein et al., 2000 classification). Many of these processes are also listed in the eight Standards for Mathematical Practice (CCSSM) Many of these processes are also listed in the eight Standards for Mathematical Practice (CCSSM)
1. Generalizing
2. Creating various specific examples for a more general statement; Creating multiple examples for conditions
3. Organizing and classifying by properties identified by students
4. Comparing and contrasting by properties identified by students
5. Organizing and classifying or comparing and contrasting by properties given by the teacher
6. Applying a problem solving strategy or mathematical knowledge to a novel problem or situation (if not obvious how it should be applied)
7. Analyzing givens, constraints, relationships, and goals of a problem or assignment; analyzing task constraints or conditions that may limit possible solution strategies or solutions

8.	Looking for and analyzing an underlying structure of mathematical objects and operations
9.	Translating a “word problem” or a “real life situation” into a mathematical representation
10.	Making sense of a mathematical representation of a word problem or real life situation in the context
11.	Flexibly using different properties of operations and mathematical objects/concepts
12.	Checking the answer or solution for a problem mathematically and whether it “makes sense”
13.	Finding errors in one’s own or others’ solutions to problems or arguments and explaining them; evaluating mathematical arguments of others (general or specific cases)
14.	Uncovering mathematical ideas behind procedures
15.	Creating and/or connecting different representation(s) for a mathematical situation
16.	Formulating students’ own definition(s) given multiple examples and non-examples of a concept
17.	Interpreting mathematical results
18.	Predicting based on analyses of data
19.	Making a connection; using an analogy
20.	Looking for patterns; conjecturing; developing a hypothesis
21.	Doing a mathematical investigation (testing a conjecture or hypothesis)
22.	Explaining, reasoning, and building a logical progression of statements to justify an argument (for a general or specific case)
23.	Proving a mathematical statement or a conjecture
24.	Identifying and analyzing assumptions for theorems or other mathematical statements
25.	Understanding and using stated assumptions, definitions, and previously established results in constructing arguments
26.	Asking questions to make sense of problems, and mathematical statements and arguments
27.	Finding a counterexample for a statement
28.	Communicating clearly, accurately, and precisely one’s own thinking orally, in writing, and using different mathematical representations
29.	Monitoring or regulating of one’s own cognitive processes or problems solving strategies

30. Using appropriate tools for solving mathematical problems or completing assignments

Insert additional questions into the following problem to increase its cognitive demand:

12. The number of dots in the figures below represent the first four *rectangular* numbers.



a. What are the first four rectangular numbers?

b. Find the next two rectangular numbers.

c. Describe the pattern of change form one rectangular number to the next.

d. Write an equation for the *n*th rectangular number *r*.

Activity 4: Homework

- Evaluate the problems that you will give to your students in the next 1-2 weeks for their level of cognitive demand.
- Specify which of the mathematical processes these problems use and which processes are lacking.
- Modify some of the problems to increase their cognitive demand using tools from today's session.

High Leverage Instructional Practices Linked to CCSS Mathematical Practices

Instruction that:

- approaches mathematics learning as problem solving (MP 1)
- emphasizes cognitively demanding conceptual tasks that encourage all students to remain engaged in the task without watering down the expectation level (maintaining cognitive demand) (MP 1)
- places the highest value on student understanding (MP 1 and 2)
- emphasizes the discussion of alternative strategies (MP 3)
- includes extensive mathematics discussion (math talk) generated through effective teacher questioning (MP 2, 3, 6, 7, and 8)
- elicits student explanations to support strategies and conjectures (MP 2 and 3)
- emphasizes the use of multiple representations (MP 4 and 5)

Reference:

Stein, M. K., Smith, M. S., Henningsen, M.A., & Silver, E.A. (2009). *Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development*. (Second Edition). New York, NY: Teachers College Press

You can find materials from this workshops here:
SERVE STEM Resources: http://www.serve.org/STEM.aspx

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